Uniform temporal convergence of numerical schemes for miscible displacement through porous media

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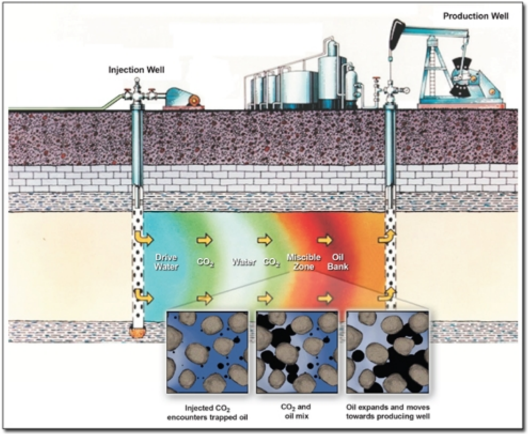
Introduction

Primary oil recovery uses natural reservoir pressure, gravity, artificial lift techniques (e.g. pumps, explosives).

When the reservoir drive is no longer sufficient to recover oil, engineers use *enhanced oil recovery* (EOR) techniques:

EOR



Source: United States Department of Energy

Outline

Peaceman model

Single-phase, miscible displacement of one incompressible fluid by another (neglecting gravity):

$$\left.\begin{aligned} \mathbf{u} &= -\frac{\mathbf{K}(x)}{\mu(c)}\nabla p \\ \operatorname{div}\mathbf{u} &= q^I - q^P \end{aligned}\right\} \quad \text{in } \Omega \times (0, T),$$

$$\partial_t c - \operatorname{div}\left(\mathbf{D}(x, \mathbf{u})\nabla c - c\mathbf{u}\right) = q^I - cq^P \quad \text{in } \Omega \times (0, T).$$

Unknowns:

- p: pressure of fluid mixture
- c: concentration (volume fraction) of injected fluid in mixture
- $\mathbf{u}$: Darcy velocity of fluid mixture

Peaceman model

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$$\partial_t c - \operatorname{div}\left(\mathbf{D}(x,\mathbf{u})\nabla c - c\mathbf{u}\right) = q^I - cq^P \quad \text{in } \Omega \times (0,T).$$

Data:

- $\mathbf{K}(x)$: absolute permeability (uniformly elliptic, bounded, matrix-valued)
- $\mu(c)$: viscosity of fluid mixture
- $\mathbf{D}(x,\mathbf{u})$: diffusion-dispersion tensor
- q^I, q^P: injection, production well source/sink terms (flow rates at wells)

Why uniform temporal convergence?

Engineers need to predict the *sweep efficiency* of the recovery process at various instants in time, and the *time to breakthrough*

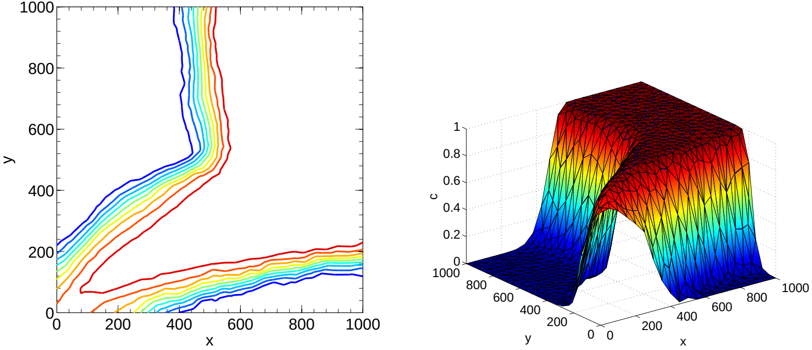

Image credit: Chainais-Hillairet, Droniou (SIAM J. Numer. Anal., 2007)

$\Rightarrow$ need a good approximation of $c(\cdot, s)$ for any $s \in (0, T)$.

$\Rightarrow$ want convergence in $L^\infty(0, T; L^2(\Omega))$.

A (very) brief history of convergence

Methods employed include

- (Mixed) Finite Elements
- (Mixed) Finite Volumes
- Discontinuous Galerkin
- Method of Characteristics
- Eulerian-Lagrangian Localised Adjoint Method

See work by J. Douglas, Jr., R.E. Ewing, T. Russell, M. Wheeler.

Examples of compactness techniques:

- J. Droniou, C. Chainais-Hillairet (SIAM J. Numer. Anal., 2007).
  Mixed Finite Volumes.
  $c_m \to c$ in $L^p(0, T; L^q(\Omega))$, for all $p < \infty$ and all $q < 2$.
- S. Bartels, M. Jensen, R. Müller (SIAM J. Numer. Anal., 2009).
  Discontinuous Galerkin.
  $c_m \to c$ in $L^2(0, T; L^2(\Omega))$.

HMM Methods

Family of methods that includes

- **H**ybrid Finite Volumes
- **M**ixed Finite Volumes
- **M**imetic Finite Differences

J. Droniou, R. Eymard, T. Gallouët and R. Herbin showed (M3AS, 2010) these 3 methods are more-or-less equivalent.

⇒ can conduct convergence analysis for Peaceman model in a reasonably abstract theoretical framework (don't need to know gritty details of methods)

HMM Scheme

In an abstract nutshell:

- Spatial mesh: $\mathcal{T} = (\mathcal{M}, \mathcal{E}) = (\text{cells}, \text{edges})$
- Temporal mesh: $0 = t^{(0)} < t^{(1)} < \cdots < t^{(N)} = T$
- Space of discrete unknowns
  $X_{\mathcal{T}} := \{c = ((c_K)_{K \in \mathcal{M}}, (c_\sigma)_{\sigma \in \mathcal{E}}) : v_K \in \mathbb{R}, v_\sigma \in \mathbb{R}\}$
- Space of discrete fluxes
  $\mathcal{F}_{\mathcal{T}} := \{F = (F_{K,\sigma})_{K \in \mathcal{M}, \sigma \in \mathcal{E}_K} : F_{K,\sigma} \in \mathbb{R}\}$
- Reconstruction operator $\Pi_{\mathcal{T}} : X_{\mathcal{T}} \to L^2(\Omega)$
- Discrete gradient operator $\nabla_{\mathcal{T}} : X_{\mathcal{T}} \to L^2(\Omega)^d$
- Discrete time derivative operator $\delta_{\mathcal{T}} : X_{\mathcal{T}} \to X_{\mathcal{T}}$

HMM Scheme

Consider sequences

$$(c^{(n)})_{n=0,\dots,N} \subset X_{\mathcal{T}}, \qquad (F^{(n)})_{n=1,\dots,N} \subset \mathcal{F}_{\mathcal{T}}$$

For $n = 1, \dots, N$,

$$c_K^{(n)} \approx c \text{ on } K \times [t^{(n-1)}, t^{(n)})$$
$$F_{K,\sigma}^{(n)} \approx -\int_\sigma \mathbf{u} \cdot \mathbf{n}_{K,\sigma} \,\mathrm{d}\gamma \text{ on } [t^{(n-1)}, t^{(n)})$$

Note the $F_{K,\sigma}^{(n)}$ come from the scheme for the pressure equation.

HMM Scheme

Find sequences $(c^{(n)})_{n=0,\dots,N} \subset X_{\mathcal{T}}$ and $(F^{(n)})_{n=1,\dots,N} \subset \mathcal{F}_{\mathcal{T}}$ such that $c^{(0)} = 0$ and for all $\varphi = (\varphi^{(n)})_{n=1,\dots,N} \subset X_{\mathcal{T}}$,

$$\int_0^T \int_\Omega \Pi_{\mathcal{T}} \delta_{\mathcal{T}} c(x,t) \Pi_{\mathcal{T}} \varphi(x,t) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \mathbf{D}(x, \mathbf{u}(x,t)) \nabla_{\mathcal{T}} c(x,t) \cdot \nabla_{\mathcal{T}} \varphi(x,t) \,\mathrm{d}x\,\mathrm{d}t$$

$$+ \sum_{n=1}^{N} \delta t^{(n-\frac{1}{2})} \sum_{K \in \mathcal{M}} \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{int}} \\ \sigma = K|L}} \left[(-F_{K,\sigma}^{(n)})^+ c_K^{(n)} - (-F_{K,\sigma}^{(n)})^- c_L^{(n)} \right] \varphi_K^{(n)}$$

$$= \int_0^T \int_\Omega \Big(q^I(x,t) - q^P(x,t) \Pi_{\mathcal{T}} c(x,t) \Big) \Pi_{\mathcal{T}} \varphi(x,t) \,\mathrm{d}x\,\mathrm{d}t,$$

where $(-F_{K,\sigma}^{(n)})^+$ and $(-F_{K,\sigma}^{(n)})^-$ denote the positive and negative parts of $-F_{K,\sigma}^{(n)}$.

Ingredients for convergence in $L^\infty(0, T; L^2(\Omega))$

- Characterisation of convergence in $L^\infty(0, T; L^2(\Omega))$
- Energy identity for continuous problem
- Estimates: energy, discrete time derivative
- Discrete Aubin-Simon compactness lemma

Convergence in $L^\infty(0, T; L^2(\Omega))$

$$\Pi_{\mathcal{T}_m} c \to c \quad \text{in } L^\infty(0, T; L^2(\Omega))$$
$$\Updownarrow$$
$$\Pi_{\mathcal{T}_m} c(T_m) \to c(T_0) \quad \text{in } L^2(\Omega) \text{ for all } T_m \to T_0.$$

$\Rightarrow$ we need

$$\int_\Omega (\Pi_{\mathcal{T}_m} c(T_m))^2 \to \int_\Omega (c(T_0))^2$$

as $m \to \infty$ (i.e. as the mesh size vanishes)

Energy identity

For any $T_0 \in (0, T)$, need solution to continuous problem to satisfy

$$\frac{1}{2}\int_\Omega c(T_0)^2 = \int_0^{T_0}\int_\Omega cq^I - \frac{1}{2}\int_0^{T_0}\int_\Omega c^2(q^I + q^P) - \int_0^{T_0}\int_\Omega |\mathbf{D}^{1/2}(\cdot,\mathbf{u})\nabla c|^2$$

- Straightforward to prove if $\mathbf{D}(x,\mathbf{u})$ is bounded...
- ...but unknown if $\mathbf{D}(x,\mathbf{u})$ grows linearly with $\mathbf{u}$ (as in practice)

Identity (*not* inequality) is critical to strengthening convergence from $L^\infty(0, T; L^2(\Omega)\text{-w})$ to $L^\infty(0, T; L^2(\Omega))$ (i.e. weak-in-space to strong-in-space).

Estimates

Standard energy estimates:

$$\|\Pi_{\mathcal{T}} c\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla_{\mathcal{T}} c\|^2_{L^2(0,T;L^2(\Omega)^d)} \leq C$$

Discrete time derivative estimate:

$$\int_0^T |\delta_{\mathcal{T}} c(t)|^4_{\star,\mathcal{T}} \,\mathrm{d}t \leq C,$$

where $|\cdot|_{\star,\mathcal{T}}$ is a discrete dual seminorm.

(Compare these to their continuous analogues)

Discrete Aubin-Simon compactness

- Sequence $(X_{\mathcal{T}_m}, \Pi_{\mathcal{T}_m}, \nabla_{\mathcal{T}_m})_{m\in\mathbb{N}}$ of discretisations
- $v_m = (v_m^{(n)})_{n=0,\ldots,N_m} \subset X_{\mathcal{T}_m}$ such that, for some $q > 1$,

$$\|\Pi_{\mathcal{T}_m} v_m\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad \int_0^T |\delta_{\mathcal{T}_m} v_m(t)|_{\star,\mathcal{T}_m}^q \,\mathrm{d}t \leq C.$$

Then $(\Pi_{\mathcal{T}_m} v_m)_{m\in\mathbb{N}}$ has a subsequence that converges in $L^\infty(0,T;L^2(\Omega)\text{-w})$, i.e. *uniformly in time* and *weakly in* $L^2(\Omega)$.

Putting it all together

Discrete Aubin-Simon implies that

$$\Pi_{\mathcal{T}_m} c(T_m) \rightharpoonup c(T_0) \quad \text{weakly in } L^2(\Omega)$$

and so

$$\liminf_{m\to\infty} \int_\Omega (\Pi_{\mathcal{T}_m} c(T_m))^2 \geq \int_\Omega (c(T_0))^2 . \tag{1}$$

Recall we want

$$\int_\Omega (\Pi_{\mathcal{T}_m} c(T_m))^2 \to \int_\Omega (c(T_0))^2 ,$$

so thanks to (1), it suffices to show

$$\limsup_{m\to\infty} \int_\Omega (\Pi_{\mathcal{T}_m} c(T_m))^2 \leq \int_\Omega (c(T_0))^2 .$$

Putting it all together

Plug in $\varphi = (c^{(1)}, \ldots, c^{(k_m)}, 0, \ldots, 0) \subset X_{\mathcal{T}}$ in the scheme, take limit superior:

$$
\begin{aligned}
\frac{1}{2} \limsup_{m\to\infty} \int_\Omega (\Pi_{\mathcal{T}_m} c(T_m))^2 \le \limsup_{m\to\infty} \int_0^{t^{(k_m)}} \int_\Omega \Pi_{\mathcal{T}_m} c q^I \\
- \frac{1}{2} \liminf_{m\to\infty} \int_0^{T_m} \int_\Omega (\Pi_{\mathcal{T}_m} c)^2 (q^I + q^P) \\
- \liminf_{m\to\infty} \int_0^{T_m} \int_\Omega \mathbf{D}(\cdot, \mathbf{u}_m) \nabla_{\mathcal{T}_m} c \cdot \nabla_{\mathcal{T}_m} c \\
= \int_0^{T_0} \int_\Omega c q^I - \frac{1}{2} \int_0^{T_0} \int_\Omega c^2 (q^I + q^P) - \int_0^{T_0} \int_\Omega |\mathbf{D}^{1/2}(\cdot, \mathbf{u}) \nabla c|^2 \\
\text{(thanks to the energy identity)} = \frac{1}{2} \int_\Omega (c(T_0))^2
\end{aligned}
$$

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