Finite Volume Approximation and Analysis of Conservation Laws Arising in Neuronal Variability

Santosh Kumar, Paramjeet Singh

School of Mathematics
Thapar University,
India.

Introduction

- It is well known that the size of presynaptic neuronal firings decide the next behavior of the neuron.
- In complex neuronal system the variability in neuronal firings is to be observed, even the external stimuli are held constant or without the external stimuli. The reason is behind that neuron possess large number of synapses and a number of inputs.

Continue...

- The neuronal activity in quantitative terms.
- In last twenty years, a number of authors present a model about the variability of neuronal firings. Their model is based on Lapicque 's integrate-and-fire-model.
- The model is based on RC circuit.
- If a short current pulse is injected into the neuron, the additional electrical charge has to go somewhere: it will charge the cell membrane. The cell membrane therefore acts like a capacitor of capacity C. Because the insulator is not perfect, the charge will, over time, slowly leak through the cell membrane. The cell membrane can therefore be characterized by a finite leak resistance R.

- If the driving current $I(t)$ vanishes , the voltage V across the capacitor is given by V_{rest}. Thus, the total current $I(t) = I_C + I_R$, I_R is resistive current passes through linear resister R and the second component I_C charges the capacitor C. Now capacity $C = Q/V$ which implies $C.V = Q$.

  Since C is fixed so we have $C\frac{d}{dt}V = \frac{d}{dt}Q = I_C$ (capacity of current).

  Now

$$R = \frac{V_R}{I_R}, \quad I_R = \frac{1}{R}(V - V_{rest}),$$

  where V_R is voltage due to resistance.
  Thus we have

$$\begin{aligned} I(t) &= C\frac{d}{dt}V + \frac{1}{R}(V - V_{rest}), \\ \tau\frac{d}{dt}V &= -(V - V_{rest}) + RI(t), \end{aligned}$$

  here $RC = \tau$, where τ is a membrane time constant, together with reset condition

$$V(t^+) = V_{reset}, \; V(t^-) = V_{thresh}.$$

  The difficulty with these models is that they have a discontinuous reset

Continue......

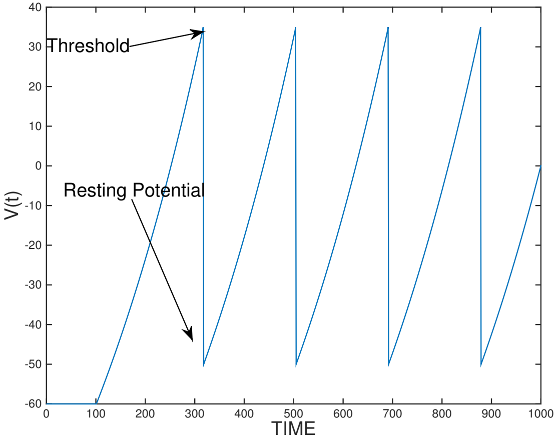

Figure : Simulation of linear integrate-and-fire model.

Quadratic integrate-and-fire model

- The quadratic integrate-and-fire model, to discuss the neuronal activity by the following equation

$$\tau \frac{dV}{dt} = C(V - V_{rest})(V - V_{thresh}) + RI(t) \quad (1)$$

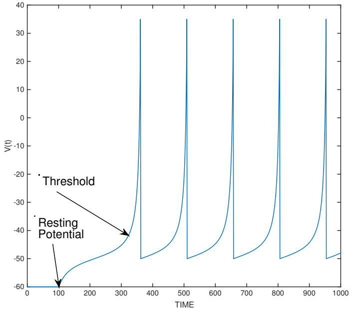

Figure : Simulation of quadratic integrate-and-fire model.

Continue......

- The equation represents the membrane potential of neuron or neuron firing at time t.
- Further the evolution in time of the density of neurons $F(v,t)$ at potential v and at time t, is determined by some assumption.
- In short time interval δt, the effect of spike (firing) at synapse by a jump of size either α or β of potential v. The jump will be positive or negative, it depends upon the frequency of excitatory neuron or inhibitory neuron respectively.
- The frequency of excitatory and inhibitory impulses are p_e and p_i per seconds, respectively.

Continue...

The change of $F(v,t)$ with time is given by

$$\begin{aligned} F(v,t+\delta t)-F(v,t) &= [1-(p_e+p_i)\delta t]\,[F(v+\delta v,t)-F(v,t)] \\ &- \underbrace{p_e\delta t[F(v,t)-F(v-\alpha,t)]}_{excitation}+\underbrace{p_i\delta t[F(v+\beta,t)-F(v,t)]}_{inhibition},\ v_{min}\leq v\leq v_{max}. \end{aligned}$$

The first term on the right hand side of above equation is taken corresponding to increase in the probability $F(v,t)$ from the resting potential during a short time interval δt, providing no impulse occur. The second term is taken for the decrease in $F(v,t)$ from quantal excitation of units between $v-\alpha$ and v while the third term for the increase in $F(v,t)$ from quantal inhibition units between v and $v+\beta$.

$$\frac{\partial F}{\partial t} = \frac{1}{\tau}\left[(C(v - v_{rest})(v - v_{thresh}) + RI(t))\right]\frac{\partial F}{\partial v} - p_e(F(v,t) - F(v-\alpha,t))$$
$$+p_i(F(v+\beta,t) - F(v,t)), \quad v_{min} \leq v \leq v_{max}.$$

The nature of the above equation depends upon the possible value of external driving current $I(t)$.

$$I(t) = I_0 + a_1 N(t)$$

where $a_1 > 0$ are constants and $N(t)$ is the mean firing rate of the neuron and given by

$$N(t) = -\int_{v_{min}}^{v_{max}} \frac{F(v,t)}{a_2} dv \tag{2}$$

where $a_2 \geq 1$ is a constant.

$$
\begin{aligned}
\frac{\partial F(v,t)}{\partial t} &- \overbrace{\frac{1}{\tau}\frac{[(C(v-v_{rest})(v-v_{thresh})+RI(t))\,F(v,t)]}{\partial v}}^{quadratic\ integrate-and-fire} \\
&= -\frac{1}{\tau}\left[C\left((v-v_{rest})+(v-v_{thresh})\right)F(v,t)\right] \\
&- \underbrace{p_e(F(v,t)-F(v-\alpha,t))}_{Excitation}+\underbrace{p_i(F(v+\beta,t)-F(v,t))}_{Inhibition}, \qquad (3) \\
&\qquad v_{min} \leq v \leq v_{max}.
\end{aligned}
$$

Numerical approximations

- Weno fifth-order finite volume approximation for spatial discretization and the evolution in time is tackled by SSP-IMEX-RK method.
- The Weno finite volume approximation is designed for hyperbolic conservation law without source term.
- We designed this scheme for source term also. We summarize the Weno procedure for our model problem.

$$\partial_t F(v,t) = -\partial_v \mathsf{H}(F,v,t) + \mathsf{S}(F,v,t). \tag{4}$$

Let $\bar{F}_j(t)$ denote the cell average of $F(.,t)$ over the cell I_j i.e.

$$\bar{F}_j(t) = \frac{1}{\Delta v} \int_{v_{j-1/2}}^{v_{j+1/2}} F(v,t)\, dv.$$

$$\frac{d}{dt}\bar{F}_j(.,t) = -\frac{1}{\Delta v}\left[\hat{\mathsf{H}}_{j+1/2} - \hat{\mathsf{H}}_{j-1/2}\right] + \mathsf{S}_j^*, \tag{5}$$

where $\hat{\mathsf{H}}_{j\pm1/2} = \mathsf{G}(F^-_{J\pm1/2}, F^+_{J\pm1/2})$ is the numerical flux that is determined by extended Roe approximate Riemann solver for monotone numerical flux function G. Moreover, the second term S_j^* is the numerical approximation of integral source term.

Weno fifth-order finite volume method

To obtain the values $F^{\pm}_{j+1/2}$ and $F^{\pm}_{j-1/2}$, we first compute these reconstructed values, i.e.

$$\hat{F}^{(r)}_{j+1/2} = \sum_{i=0}^{2} c_{ri}\ \bar{F}_{j-r+i}, \quad r = 0, 1, 2$$

corresponding to three different stencil $S_r(j) = \{v_{j-r}, v_{j-r+1}, v_{j-r+2}\}, \quad r = 0, 1, 2$. Similarly we can find the values $\hat{F}^{(r)}_{j-1/2}$.

The coefficients c_{ri} is obtained by the following expression

$$c_{ri} = \sum_{m=i+1}^{3} \frac{\sum\limits_{l=0,\ l\neq m}^{3} \prod\limits_{q=0,\ q\neq m,l}^{3} (r-q+1)}{\prod\limits_{l=0,\ l\neq m}^{3} ((m-l)}, \quad i=-1,0,1,2$$

which does not depend on j or mesh length Δv. Now we would take a convex combination of the values $\hat{F}^{(r)}_{j+1/2}$ and $\hat{F}^{(r)}_{j-1/2}$ to find a new approximation $F^{\pm}(v_{j+1/2})$ and $F^{\pm}(v_{j-1/2})$, respectively.

$$F^{\pm}(v_{j+1/2}) = \sum_{r=0}^{2} w_r^{\pm} \hat{F}^{(r)}_{j+1/2}, \quad \sum_{r=0}^{2} w_r^{\pm} = 1,$$

$$F^{\pm}(v_{j-1/2}) = \sum_{r=0}^{2} \tilde{w}_r^{\pm} \hat{F}^{(r)}_{j-1/2}, \quad \sum_{r=0}^{2} \tilde{w}_r^{\pm} = 1.$$

The non-linear weights $w_r^{\pm}$ are given by

$$w_r^{\pm} = \frac{\alpha_r^{\pm}}{\sum_{m=0}^{2} \alpha_m^{\pm}}, \quad \alpha_r^{\pm} = \frac{\gamma_r}{(\epsilon + \beta_r^{\pm})^2}, \quad r = 0, 1, 2$$

- Here γ_r are the linear weights which yield 5-th order accuracy and its value are $\gamma_0 = 1/10,\ \gamma_1 = 6/10,\ \gamma_2 = 3/10$ in fifth-order weno method.
- β_r are the smoothness indicator of the stencil $S_r(j)$ which measure the smoothness of the function $F(v,t)$ in the stencil.
- ϵ is small constant used to avoid the denominator become zero. Now the value of smoothness indicator are given by the relation

$$\beta_r^{+} = \sum_{m=1}^{2} \int_{v_{j-1/2}}^{v_{j+1/2}} \Delta v^{2l-1} \left(\frac{\partial^m q_r(v)}{\partial^m v} \right)^2 dv$$

- $q_r(v)$ is the reconstruction polynomial on the stencil $S_r(j) = \{v_{j-r}, v_{j-r+1}, v_{j-r+2}\}, \quad r = 0, 1, 2.$

Similarly other non-linear weights $\tilde{w}_r^{\pm}$ are given by

$$\tilde{w}_r^{\pm} = \frac{\tilde{\alpha}_r^{\pm}}{\sum_{m=0}^{2} \tilde{\alpha}_m^{\pm}}, \quad \tilde{\alpha}_r^{\pm} = \frac{\tilde{\gamma}_r}{(\epsilon + \beta_r^{\pm})^2}, \tilde{\gamma}_r = \gamma_{2-r}, \quad r = 0, 1, 2.$$

- After the values $F^{-}_{j+1/2}$, $F^{+}_{j+1/2}$ are determined, we can used the Roe Riemann solver to find the value of flux function $\mathsf{G}(F^{-}_{j+1/2}, \; F^{+}_{j+1/2})$.

$$
\begin{aligned}
\mathsf{G}(F^-_{j+1/2}, F^+_{j+1/2}) &= \frac{1}{2}\left(\mathsf{H}(F^-_{j+1/2}, v_{j+1/2}) + \mathsf{H}(F^+_{j+1/2}, v_{j+1/2})\right) \\
&- \frac{1}{2}\left(\left|\tilde{\lambda}_{j+1/2}\right|\left(F^+_{j+1/2} - F^-_{j+1/2}\right) + sgn\left(\tilde{\lambda}_{j+1/2}\right)\mathsf{V}_{j+1/2}\right).
\end{aligned}
$$

- The next step is to extend WENO-FVM scheme to the solution of our source term.
- The source term is evaluated by simple point-wise approximation, but in many cases this approach gives poor result.
- Upwinding include in the approximation of source term which improve the stability of scheme and shown by Roe. The decomposition of the source term is of the form.

$$\mathsf{S}_j^* = \mathsf{S}_{j-\frac{1}{2},R} + \mathsf{S}_{j+\frac{1}{2},L} + \mathsf{S}_{j,C}.$$

$$
\begin{aligned}
\mathsf{S}_{j-\frac{1}{2},R} &= \frac{1+sgn(\tilde{\lambda}_{j+1/2})}{2}\mathsf{S}_{j-\frac{1}{2}}; \quad \mathsf{S}_{j-\frac{1}{2}} = \mathsf{S}(F^{\pm}_{j-1/2}, v_{j-1/2}) \\
\mathsf{S}_{j+\frac{1}{2},L} &= \frac{1-sgn(\tilde{\lambda}_{j+1/2})}{2}\mathsf{S}_{j+\frac{1}{2}}; \quad \mathsf{S}_{j+\frac{1}{2}} = \mathsf{S}(F^{\pm}_{j+1/2}, v_{j+1/2}) \\
\mathsf{S}_{j,C} &= \mathsf{S}\left(F^{+}_{j-1/2}, F^{-}_{j+1/2}\right).
\end{aligned}
$$

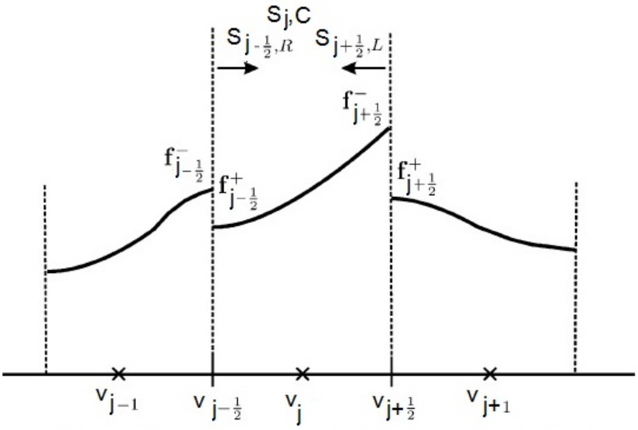

Figure : Numerical flux and source terms for the finite volume scheme.

Temporal discretiztaion

- To solve hyperbolic PDEs, the common approach is to first discretize the spatial variables to get a semi discrete method of line of approximation. In this approach we have time-dependent ODE system after reducing the original PDE. Then the ODE system is solved by some appropriate ODE solver, but the main concern is about its stability.
- A linear stability analysis is not sufficient for hyperbolic PDE in which shocks and discontinuous solutions generally occur. Therefore we need a strong measure of stability for the solution of hyperbolic PDEs.
- We used a strong-stability-preserving Runge-Kutta (SSPRK) for the advection flux term and then implicit-explicit Runge-Kutta (IMEX-RK) method used for source term.

- The semi-discrete scheme of Eqs. (4) at n-th time level

$$\frac{d\bar{F}_j^n}{dt} = \mathsf{L}(\bar{F}_j^n) + \mathsf{S}^*(\bar{F}_j^n), \tag{6}$$

where $\mathsf{L}(\bar{F}_j^n)$ is the numerical approximation of advection flux obtained by WENO finite volume approximation (WENO-FVM) i.e.

$$\mathsf{L}(\bar{F}_j^n) = -\frac{1}{\Delta v}\left[\mathsf{G}(F_{j+1/2}^-, F_{j+1/2}^+) - \mathsf{G}(F_{j-1/2}^-, F_{j-1/2}^+)\right].$$

While $\mathsf{S}^*(\bar{F}_j^n)$ is the approximation of source term determined by WENO-FVM.

- We now compute the numerical solution from n-th to $(n+1)$-th time level. For this we use SSP-RK method which leads to the following approximation

$$
\begin{aligned}
\bar{F}_j^{(0)} &= \bar{F}_j^n \\
\bar{F}_j^{(k)} &= \sum_{l=0}^{k-1} \left[\alpha_{kl} \bar{F}_j^{(l)} + \beta_{kl} \Delta t \left(\mathsf{L}(\bar{F}_j^{(l)}) + \mathsf{S}^*(\bar{F}_j^{(l)}) \right) \right], \; k = 1 \ldots m \\
\bar{F}_j^{(m)} &= \bar{F}_j^{n+1}.
\end{aligned} \tag{7}
$$

- $$\begin{aligned} \bar{F}_j^{(0)} &= \bar{F}_j^n \\ \bar{F}_j^{(k)} &= \bar{F}_j^{(0)} + \Delta t \sum_{l=0}^{k-1} D_{kl} \left(\mathsf{L}(\bar{F}_j^{(l)}) + \mathsf{S}^*(\bar{F}_j^{(l)}) \right), \; k = 1 \ldots m \\ F^{(m)} &= \bar{F}_j^{n+1}. \end{aligned}$$

Implicit-Explicit-Runge-Kutta (IMEX-RK):

The scheme is used for balance laws, It takes the form

$$
\begin{aligned}
\bar{F}_j^{(l)} &= \bar{F}_j^n + \Delta t \sum_{i=1}^{l-1} \tilde{a}_{li} \mathsf{L}(\bar{F}_j^{(i)}) + \Delta t \sum_{i=1}^{N} a_{li} \mathsf{S}^*(\bar{F}_j^i) \\
\bar{F}_j^{n+1} &= \bar{F}_j^n + \Delta t \sum_{l=1}^{N} \tilde{\omega}_l \mathsf{L}(\bar{F}_j^{(l)}) + \Delta t \sum_{l=1}^{N} \omega_l \mathsf{S}^*(\bar{F}_j^l).
\end{aligned}
\tag{8}
$$

The above scheme can be written with double tableau in Butchar form

$$\begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{\omega}^T \end{array} \qquad \begin{array}{c|c} c & \mathsf{A} \\ \hline & \omega^T \end{array}$$

- $\tilde{A} = (\tilde{a}_{li})\,, \tilde{a}_{li} = 0$ for $i \geq l$ and $A = (a_{li})$ are $N \times N$ matrices. The above scheme is explicitly in L and implicitly in S.

By using the above scheme, we present the algorithm that updates the numerical solution in following manner.

- Explicit flux function

$$\bar{F}_{j*}^{(l)} = \bar{F}_j^n + \Delta t \sum_{i=1}^{l-2} \tilde{a}_{li} \mathsf{L}(\bar{\mathsf{F}}_{\mathsf{j}}^{(\mathsf{i})}) + \Delta t \tilde{a}_{l,l-1} \mathsf{L}(\bar{F}_j^{(l-1)}).$$

- Implicit source function

$$\bar{F}_j^{(l)} = \bar{F}_{j*}^{(l)} + \Delta t \sum_{i=1}^{l-1} a_{li} \mathsf{S}(\bar{F}_j^{(i)}) + \Delta t a_{ll} \mathsf{S}(\bar{F}_j^{(l)}).$$

- Final solution at next time level

$$\bar{F}_j^{n+1} = \bar{f}_j^n + \Delta t \sum_{l=1}^{N} \tilde{\omega}_l \mathsf{L}(\bar{F}_j^{(l)}) + \Delta t \sum_{l=1}^{N} \omega_l \mathsf{S}(\bar{F}_j^l).$$

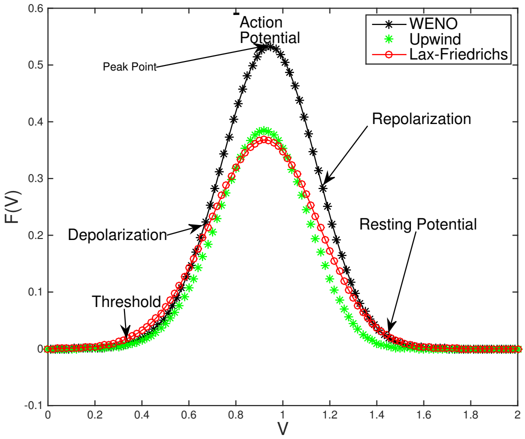

Figure : Time evolution of firing rate and comparison between WENO, upwind and Lax-Friedrichs scheme

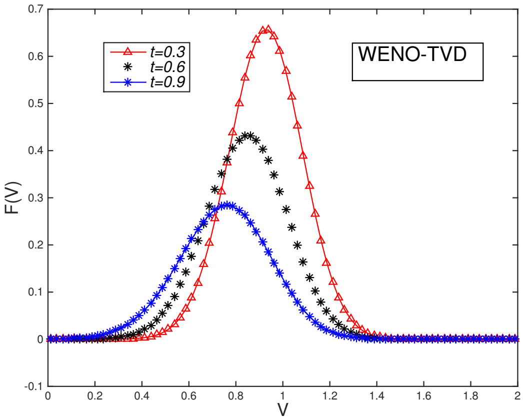

Figure : WENO simulation for the time evolution of solution $F(v,t)$ at different instant t=0.3, t=0.6, t=0.9

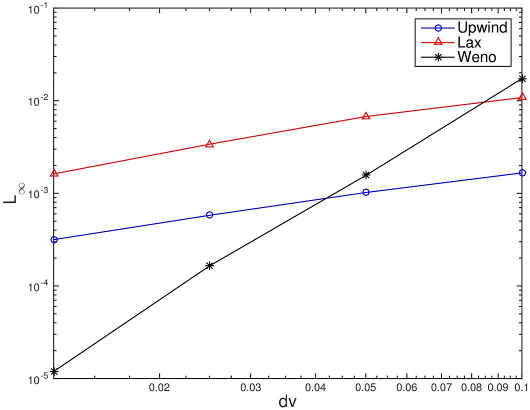

Figure : Error in L_∞ norm with WENO, upwind & Lax-Friedrichs

References I

L. Lapicque, *Recherches quantitatives sur l'excitation électrique des nerfs traitée comme une polarization*. J. Physiol. Pathol. Gen. 9 (1907), 620–635.

Richard B. Stein, *A theoretical analysis of neuronal variability* , Biophysic. J., 5, 173-194, 1965

Kapil K. Sharma and Paramjeet Singh, *Hyperbolic partial differential-difference equation in the mathematical modeling of neuronal firing and its numerical solution*, Appl. Math. Comput. 201, no.1-2, 229-238, 2008

Khashayar Pakdaman, *Periodically forced leaky integrate-and-fire model*, Physical Review E. 63, 041907, 2001.

J.Touboul, *Importance of the cutoff value in the quadratic adaptive integrate -and -fire model*, Neural Computation. 21, 2114-2122, 2009

P. Kramer K. Newhall, G. Kovacic, *Dynamics of current -based, Poisson driven ,integrate- and-fire neuronal networks.* , Commutation in Mathematical Sciences. 8, 541-600, 2010

Sigal Gottlieb and Chi -Wang -Shu, *Total variations diminishing runge kutta schemes*, Mathematics of Computation. 67, no.221 , 73-85, 1998

Chi-Wang- Shu, *Total- variation- diminishing time discretization* , SIAM J. 9, no.6, 1998

References II

Joël Pham, Khashayar Pakdaman, *Jean Champagnat and Jean-François Vibert*, Activity in sparsely connected excitatory neural networks: effect of connectivity. Neural Networks 11, 415-434, 1998.

Khashayar Pakdaman, Benoît Perthame and Delphine Salort, *Dynamics of a structured neuron poplution*, Nonlinearity, 23, 55-75, 2010.

Fadia Bekkal Brikci, Jean Clairambault, Benjamin Ribba and Benoît Perthame, *An age-and-cyclin-structured cell population model for healthy and tumoral tissues*, J. Math. Biol. 57, no. 1, 91-110, 2008.

R. J. Leveque, Numerical Methods For Conservation Laws, Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel, 1999.

C. Grossmann, H. G. Roos and M. Stynes, *Numerical Treatment of Partial Differential Equations*, Springer-Verlag, Berlin, 2007.

E. Godlewski, P. A. Raviart, *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Springer, New York, 1996.

References III

S. Gottlieb, C. W. Shu, *Total Variations Diminishing Runge Kutta Schemes*, Math. Comp. 67, no. 221, 73-85, 1998.

G.S. Jiang, C. W. Shu, *Efficient Implementation of Weighted ENO Schemes*, J. Comput. Phys. 126, 202-228,1996.

THANK YOU VERY MUCH FOR YOUR ATTENTION...